The Continuing Story of Zeta

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1. TAKING THE LOW ROAD. Riemann's Zeta Function $\zeta(s)$ is defined for complex $s=\sigma+it$ with $\Re(s)=\sigma>1$ by the formula

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

There are many ways to obtain the analytic continuation of $\zeta(s)$ to the left hand half-plane. The high road, Riemann's own [10], uses contour integration at an early stage, and leads directly to the functional equation. Many authors ([1], [3], [4], [8], [9], [12], and [13]) use this method, or variants of it, often at a more leisurely pace. Other methods are known (Chapter 2 of [12] lists seven) but a toll seems inevitable on any route ending with the functional equation.

There are lower roads which give both the continuation to the whole plane and the evaluation at non-positive integers but stop short of proving the functional equation. If these are rigorous, yet quick and simple, there must surely be a case for using them as well. The point of this article to draw wider attention to these, often very scenic, roads. In his beautiful article [2, Sect. 7], Ayoub comments upon Euler's paper of 1740 in which he boldly evaluates divergent series to obtain $\zeta(-k)$ for integers $k\geqslant 0$, thereby predicting the functional equation. Recently, Sondow [11] has noted one way in which Euler's argument can be made rigorous. Simultaneously, Mináč [6] showed how to evaluate $\zeta(-k)$ in an extremely simple and elegant way, by integrating a polynomial on $[0,1]$. More recently, Murty and Reece [7] have shown how the continuation and evaluation of the Hurwitz zeta function can be obtained in a simple down-to-earth way and this is applicable to $\zeta(s)$ and many L-functions. The point of this note is to highlight just how easily the continuation and evaluation of $\zeta(s)$ can be obtained. All that we say can be found in the articles cited. For example, our work-horse (10) is the truncation of Landau's formula [5, p. 274].

2. A JOURNEY OF A THOUSAND MILES... Notice that for $\sigma>1$,

$$\int_1^{\infty}x^{-s}\,dx=\frac{-1}{1-s}=\frac{1}{s-1},$$

which yields at once the continuation to the whole complex plane of the function represented by the integral for $\sigma>1$. Obviously the continuation is analytic everywhere apart from a simple pole at $s = 1$. For $\sigma > 1$,

$$\begin{aligned}
\frac{1}{s-1} &= \int_1^\infty x^{-s}\,dx = \sum_{n=1}^\infty \int_n^{n+1} x^{-s}\,dx \\
&= \sum_{n=1}^\infty \int_0^1 (n+x)^{-s}\,dx = \sum_{n=1}^\infty \frac{1}{n^s}\int_0^1 \left(1+\frac{x}{n}\right)^{-s} dx. \qquad (1)
\end{aligned}$$

All the sums converge absolutely for $\sigma > 1$. In what follows we assume that $\sigma > 1$ and that $|s|$ is bounded by K, a fixed (although arbitrary) constant. Now begin the binomial expansion of the bracketed term, noting that the higher binomial coefficients all include a factor s:

$$\left(1+\frac{x}{n}\right)^{-s} = 1 - \frac{sx}{n} + sE_1(s,x,n), \qquad (2)$$

where the function E_1 satisfies

$$|E_1(s,x,n)| \leqslant \frac{C_1 x^2}{n^2} \leqslant \frac{C_1}{n^2}, \qquad (3)$$

for all $x \in [0,1]$ and all $n \geqslant 1$, with $C_1 = C_1(K)$ (since E_1 is just the error term of a Taylor series in x/n). Substitute Equation (2) into the sum (1) and perform the integration with respect to x. We find that

$$\frac{1}{s-1} = \zeta(s) - \frac{s}{2}\zeta(s+1) + sA_1(s), \qquad (4)$$

where $A_1(s)$ is analytic for $\sigma > -1$ by (3). Thus Equation (4) may be used to extend $\zeta(s)$ to the half-plane $\sigma > 0$. It even shows that the extended function will be analytic there apart from a simple pole at $s = 1$ with residue 1. In other words, Equation (4) implies that

$$\lim_{s\to 1}(s-1)\zeta(s) = 1. \qquad (5)$$

Equation (5) can also be written $\lim_{s\to 0} s\zeta(s+1) = 1$. Using this fact, and letting $s \to 0^+$ in Equation (4), we obtain

$$-1 = \zeta(0) - \frac{1}{2},$$

which yields the known value $\zeta(0) = -1/2$.

The preceding argument begins with the binomial estimate (2), finds the analytic continuation of the zeta function to the half-plane $\sigma > 0$ and evaluates $\zeta(0)$. What happens if more terms of the binomial expansion are included? An additional term in the binomial expansion gives

$$\left(1+\frac{x}{n}\right)^{-s} = 1 - \frac{sx}{n} + \frac{s(s+1)x^2}{2n^2} + (s+1)E_2(s,x,n);$$

notice that the higher binomial coefficients all include a factor $(s+1)$. Here, E_2 is a function which satisfies

$$|E_2(s,x,n)| \leqslant \frac{C_2 x^3}{n^3} \leqslant \frac{C_2}{n^3},$$

for all $x \in [0,1]$ and all n, where $C_2 = C_2(K)$. Substituting this into (1) and integrating as before gives

$$\frac{1}{s-1} = \zeta(s) - \frac{s}{2}\zeta(s+1) + \frac{s(s+1)}{6}\zeta(s+2) + (s+1)A_2(s), \tag{6}$$

where $A_2(s)$ is analytic for $\sigma > -2$. Thus, Equation (6) may be used to continue $\zeta(s)$ to the half-plane $\sigma > -1$. As before, letting $s \to -1^+$ and using Equation (5), we obtain

$$-\frac{1}{2} = \zeta(-1) + \frac{1}{2}\zeta(0) - \frac{1}{6} = \zeta(-1) - \frac{1}{4} - \frac{1}{6}$$

yielding the known value $\zeta(-1) = -1/12$.

3. GENERAL METHOD. This method can be repeated in order to continue $\zeta(s)$ further and further to the left of the complex plane. Moreover, it yields the explicit evaluation at the non-positive integers in terms of the *Bernoulli numbers*. The sequence of *Bernoulli numbers* (B_n) is defined via the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{7}$$

from which it is clear that all the B_n are rational numbers. We need two well-known properties of this fascinating sequence which are stated in the following lemma.

Lemma 3.1. *With B_n defined by (7),*

$$\sum_{n=0}^{N-1} \binom{N}{n} B_n = 0 \qquad \text{for all } N > 1, \tag{8}$$

and

$$B_n = 0 \qquad \text{for all odd } n \geqslant 3.$$

Proof. The relation (7) can be written

$$(e^x - 1)\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = x.$$

For $N > 1$ the coefficient of x^N in the left-hand side is

$$\sum_{m=0}^{N-1} \frac{1}{(N-m)!m!} B_m,$$

which gives (8) after multiplying by $N!$. The second statement follows from the fact that

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x(1 + e^x)}{e^x - 1}$$

is an even function. $\square$

The recurrence relation (7) can be used to calculate B_n inductively. The first few Bernoulli numbers are given below.

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

Theorem 3.2. *There is an analytic continuation of $\zeta(s)$ to the entire complex plane where it is analytic apart from a simple pole at $s = 1$ with residue 1. For all $k \geqslant 1$,*

$$\zeta(-k) = -\frac{B_{k+1}}{k+1}. \tag{9}$$

Note that Equation (9) is not true for $k = 0$ but our method has already given us the special value $\zeta(0) = -1/2$.

Proof of Theorem 3.2. The analytic continuation of the zeta function to the half-plane $\sigma > -k$ arises in exactly the same way as before, by extracting an appropriate number of terms of the binomial expansion and using induction. For integral $k \geqslant 0$ and $\sigma > 1$, this gives the relation

$$\begin{aligned} \frac{1}{s-1} &= \zeta(s) + \sum_{r=0}^{k} \frac{(-1)^{r+1} s(s+1)\dots(s+r)}{(r+2)!} \zeta(s+r+1) \\ &\qquad + (s+k) A_{k+1}(s) \end{aligned} \tag{10}$$

where $A_{k+1}(s)$ is analytic in $\sigma > -(k+1)$, again because all higher binomial coefficients include a factor $(s+k)$. Notice that $k = 0$ gives Equation (4) and $k = 1$ gives Equation (6).

By induction, we may assume that $\zeta(s)$ has already been extended to the half-plane $\sigma > 1 - k$ so Equation (10) is valid there, because the singularities at $s = 0, -1, \dots$ are removable. All the functions in Equation (10) except $\zeta(s)$ are defined at least for $\sigma > -k$, which gives the analytic continuation of $\zeta(s)$ to that half-plane. Let $s \to -k^+$ in (10) and use Equation (5) for the term with $r = k$ to obtain

$$-\frac{1}{k+1} = \zeta(-k) + \sum_{r=0}^{k-1} \binom{k}{r+1} \frac{\zeta(-k+r+1)}{r+2} - \frac{1}{(k+1)(k+2)}.$$

Writing r for every $r + 1$ simplifies this to

$$0 = \zeta(-k) + \frac{1}{k+2} + \sum_{r=1}^{k} \binom{k}{r} \frac{\zeta(-k+r)}{r+1}.$$

The term with $r = k$ is known. Using induction on the others gives

$$0 = \zeta(-k) + \frac{1}{k+2} - \sum_{r=1}^{k-1} \binom{k}{r} \frac{B_{k-r+1}}{(r+1)(k-r+1)} - \frac{1}{2(k+1)}. \tag{11}$$

A simple manipulation of factorials gives

$$\frac{(k+1)(k+2)}{(r+1)(k-r+1)} \binom{k}{r} = \binom{k+2}{r+1} = \binom{k+2}{k-r+1},$$

which transforms Equation (11) to

$$0 = \zeta(-k) + \frac{k}{2(k+1)(k+2)} - \frac{1}{(k+1)(k+2)} \sum_{r=1}^{k-1} \binom{k+2}{k-r+1} B_{k-r+1}. \tag{12}$$

Multiply by $(k+1)(k+2)$ and apply Equation (7) with $N = k+2$. Only the terms for $r = 0, k, k+1$, missing in Equation (12) survive, yielding

$$\begin{aligned} 0 &= (k+1)(k+2)\zeta(-k) + \frac{k}{2} + (k+2)B_{k+1} + (k+2)B_1 + B_0 \\ &= (k+1)(k+2)\zeta(-k) + (k+2)B_{k+1} \end{aligned}$$

and this completes the induction argument. □

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